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SEARCH FOR A FIXED TARGET BY A MOVING OBJECT*

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The problem of searching for a fixed target by a controlled object whose motion is governed by a system of ordinary differential equations or by a linear discrete system with a given probability density distribution of the initial position. The necessary conditions for the optimality of the control which maximizes the probability that the object's trajectory will reach the given target set after a fixed time, are determined.

1. The continuous case. Let the dynamics of the controlled object be specified by the system

$$z^{\cdot} = f(z, t, u),\ z \in E^n;\ f(z, t, u) = \operatorname{col}(f_i, (z, t, u)),\ i = 1, 2, \ldots, n \tag{1.1}$$

and let the assumptions ensuring the existence, continuability and uniqueness of the solution all hold /1/ The probabilistic distribution of the position of the object at the initial instant t_0 of density $p_0(z)$ is given. A control $u(t), t \in [t_0, T]$ is sought in the class Ω of measurable functions with values from the set U, which gives the maximum probability that the trajectory of the system (1.1) will reach the given set $M, M \subset E^n$ at a fixed finite instant of time T.

We denote the distribution probability density of the object at the time t by $p(t, z)$, and the maximizing probability by P. We have the formula

$$P(T, M) = \int_M p(t, y)\, dy \tag{1.2}$$

The density $p(t, z)$ satisfies the partial differential equation (the Fokker-Planck-Kholmogorov equation) /2/ with initial conditions

$$\partial p(t, z)/\partial t = -(\nabla, f(z, t, u)\, p(t, z));\ p(0, z) = p_0(z),\quad \nabla = (\partial/\partial z_1, \ldots, \partial/\partial z_n) \tag{1.3}$$

Equation (1.3) holds for any admissbile control $u(\cdot) \in \Omega$ and is a quasilinear first-order equation with $n+1$ independent variables. Its integration is equivalent /3/ to integrating the set of ordinary differential equations (1.1), and the equation

$$dp(t, z)/dt = -p(t, z)(\nabla, f(z, t, u)) \tag{1.4}$$

Let us denote by $z(t, y)$ the curve representing the solution of (1.1) and passing through the point y at the time T, i.e. $z(T, y) = y$. We substitute the solution obtained into (1.4), and solve it for the initial condition

$$p(t_0, z(t_0, y)) = p_0(z(t_0, y)) \tag{1.5}$$

Substituting the solution of problem (1.4), (1.5) into (1.2), we obtain

$$P(T, M) = \int_M p_0(z(t_0, y)) \exp\left\{-\int_{t_0}^{T} (\nabla, f(z(\theta, y), \theta, u(\theta)))\, d\theta\right\} dy \tag{1.6}$$

If the right hand side of (1.1) is independent of the phase variable z, then formula (1.6) simplifies considerably (the exponential term becomes equal to unity).

Let us investigate in more detail the case of the linear dynamics of the moving object

$$z^{\cdot} = Az + u\quad \left((\nabla, f(z, t, u)) = \sum_{i=1}^{n} a_{ii} = \operatorname{tr} A\right) \tag{1.7}$$

where A is a square $n \times n$-matrix. Having written the solution of (1.7) in accordance with Cauchy's formula under the condition $z(T, y) = y$ and for the case when $t_0 = 0$, we write (1.6) in the form

$$P(T, M) = f_0(u) = \exp(-T \operatorname{tr} A) \int_M p_0\Big(\exp(-AT)\, y - \int_0^T \exp(-A\theta)\, u(\theta)\, d\theta\Big)\, dy \tag{1.8}$$

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We assume that the density $p_0(z)$ is a continuously differentiable function, the set M is convex and closed, and U is a convex compactum. Then the set Ω is also convex.

Before formulating the results, we introduce some necessary notation. The quantity $\nabla_z f(z)$ will denote the vector gradient of the differentiable function $f(z)$. For a convex set X the symbol $K_X(x_0)$ will denote the cone of possible directions at the point $x_0 \in X$, i.e. a convex cone composed of the vectors $e \in X$ such that $x(\lambda) = x_0 + \lambda e \in X$ for all λ, $0 \leqslant \lambda \leqslant \lambda_1$ for sufficiently small λ_1. We denote the transpose of the matrix A by A^T.

The theorem given below results from the application of the necessary conditions /4/ to the problem of maximizing the probability (1.8).

Theorem 1. Let $u_0(\cdot) \in \Omega$ be a control ensuring that the function $f_0(u)$ has a maximum. Then the following inequality will necessarily hold:

$$\int_0^T \Big(\exp(-A^T\theta) \int_M \nabla_z p_0\Big(\exp(-AT)\, y - \int_0^T \exp(-A\theta_1)\, u_0(\theta_1)\, d\theta_1\Big) dy, \Delta u\Big) d\theta \geqslant 0, \quad \forall \Delta u \in K_\Omega(u_0)$$

Proof. Since

$$\max_{u \in \Omega} f_0(u) = -\min_{u \in \Omega}(-f_0(u))$$

the problem of maximizing the probability (1.8) is reduced to that of minimizing the function $f_0(u)$ over $u \in \Omega$.

The function $f_0(u)$ represents, apart from a positive coefficient, a superposition of the function $f(x)$ defined on E^n and the operator $\mathbf{A}u$, where

$$f(x) = \int_M p_0(\exp(-AT)\, y - x)\, dy, \quad \mathbf{A}u = \int_0^T \exp(-A\theta)\, u(\theta)\, d\theta$$

The function $f(x)$ is continuously differentiable and satisfies the Lipshitz condition within the bounded domain of its definition by virtue of the assumptions concerning the function $p_0(x)$ and the set U. The operator $\mathbf{A}u: L_\infty[0, T] \to E^n$ is a Gateaux-differentiable operator (here $L_\infty[0, T]$ is a space of measurable bounded functions defined on $[0, T]$ and representing a complete normed (Banach) space /5/).

In this case we can use Theorem 3.1 of /4/ on the superposition of a quasidifferentiable function and a Gateaux-differentiable operator. According to this theorem the functions $f(\mathbf{A}u)$ and $f_0(u)$ are quasidifferentiable and satisfy the Lipshitz condition. Therefore the results obtained in /4/ can be used, implying that the necessary conditions for the function $f_0(u)$ to have a minimum at the point u_0 have the form of an inequality

$$f_0'(u_0, \Delta u) \leqslant 0 \tag{1.9}$$

which must hold for all $\Delta u \in K_\Omega(u_0)$. Here $f_0'(u_0, \Delta u)$ is a derivative of $f_0(u)$ in the direction Δu at the point u_0.

The same theorem implies that

$$f_0'(u_0, \Delta u) = (\mathbf{A}_0')^* \nabla_x f(\mathbf{A}u_0)(\Delta u) \tag{1.10}$$

where $\mathbf{A}_0'$ is the Gateaux derivative of the operator $\mathbf{A}u$ at the point u_0, $\mathbf{A}$ is an operator conjugate to $\mathbf{A}_0'$, $\nabla_x f(\mathbf{A}u_0)(\cdot)$ is a linear functional assuming at every $x \in E^n$ the value $(\nabla_x f(\mathbf{A}u_0), x)$, $\mathbf{A}u$ is a linear bounded operator and $\mathbf{A}_0' = \mathbf{A}$. The operator $\mathbf{A}^*$ places in 1:1 correspondence to every functional $\varphi(x) = (\varphi, x)$ defined on E^n, $\varphi \in E^n$, a functional $\mathbf{A}^*\varphi$ defined on $L_\infty[0, T]$ such that $\mathbf{A}^*\varphi(u) = \varphi(\mathbf{A}u)$. Taking into account the concrete form of the operator $\mathbf{A}u$ we obtain, after some reduction,

$$\mathbf{A}^*\varphi(\Delta u) = \int_0^T (\exp(-A^T\theta)\, \varphi, \Delta u(\theta))\, d\theta$$

The operator $\mathbf{A}^*$ is applied in (1.10) to the functional $\nabla_x f(\mathbf{A}u_0)$, therefore

$$\mathbf{A}^*\nabla_x f(\mathbf{A}u_0)(\Delta u) = -\int_0^T \Big(\exp(-A^T\theta) \int_M \nabla_z p_0\Big(\exp(-AT)\, y - \int_0^T \exp(-A\theta_1)\, u_0(\theta_1)\, d\theta_1\Big) dy, \Delta u(\theta)\Big) d\theta \tag{1.11}$$

The formulas (1.8)–(1.11) yield the proof of the theorem.

Example. Let us consider the simple motion in a plane

$$y^{\cdot}=u,\quad y=(y_1,y_2)^T,\quad y_0=(y_0',y_0'')^T,\quad M=\{y\colon \|y-y_0\|\leqslant\varepsilon\},\quad \|u\|\leqslant a$$

The initial distribution is normal $N(0,1)$ and of density

$$p_0(y)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_1^2+y_2^2}{2}\right)$$

The optimizing probability has the form

$$P(T,M)=\iint\limits_{y_0+S_0(\varepsilon)}\exp\left\{-\frac{1}{2}[(y_1-Tu_1)^2+(y_2-Tu_2)^2]\right\}dy_1\,dy_2 \tag{1.12}$$

where $S_0(\varepsilon)$ is a sphere of radius ε with centre at the origin of coordinates.

The necessary condition for the control $u_0=(u_0^1,u_0^2)^T$ to be optimal is

$$\iint\limits_{S_0(\varepsilon)}\exp\left\{-\frac{1}{2}[(y_1+y_0'-Tu_0^1)^2+(y_2+y_0''-Tu_0^2)^2]\right\}\times$$
$$[(y_1+y_0'-Tu_0^1)\Delta u_1+(y_2+y_0''-Tu_0^2)\Delta u_2]\,dy_1\,dy_2\geqslant 0$$

and since the domain of integration is symmetrical, it is sufficient for the condition to be satisfied at the origin of coordinates. The condition is satisfied by the control u_0 when $u_0^1=y_0'/T$, $u_0^2=y_0''/T$ when $T\geqslant\|y_0\|/a$.

What we just said agrees with the following geometrical considerations. The integral in (1.12) represents the volume of a body cut from a cylinder by the surface $z=p_0(y)$, with the circle of radius ε serving as the base, and will be the largest when the centre of this circle is placed at the origin of coordinates.

2. The discrete case. The motion of the target is governed by the system of difference equations

$$x_{k+1}=Ax_k+u_k,\quad x_k\in E^n \tag{2.1}$$

where x_k is the position of the object at the k-th step, u_k is the control chosen from the set U at the k-th step, U is a convex compact and A is a $n\times n$-matrix. The probabilistic distribution of the position of the target at the initial instant, of density $p_0(x)$, is given.

We choose the controls $u_1,\ldots,u_k$ so as to maximize the probability of emergence of the target after k steps at the prescribed set M, $M\subset E^n$.

According to (2.1), when $u_1,\ldots,u_k$ are fixed, then the position of the target at the k-th step is given in terms of its initial position as follows:

$$x_k=A^kx_0+\sum_{i=1}^{k}A^{k-i}u_i$$

This yields, assuming that the matrix A^{-1} inverse to A exists,

$$P(x_k\in M)=\int\limits_{\Lambda(M)}p_0(z)\,dz,\quad \Lambda(M)=(A^{-1})^kM-\sum_{i=1}^{k}(A^{-1})^iu_i$$

Let us make the change of variables $z=\Lambda(z_1)$ in the integrand, and introduce the following vector and a matrix:

$$u=\operatorname{col}(u_1,\ldots,u_k),\quad A_k=(A^{-1}\ \ldots\ (A^{-1})^k)$$

The expression for the probability $P(x_k\in M)$ now becomes

$$P(x_k\in M)=|A^{-1}|^k\int\limits_M p_0((A^{-1})^kz-A_ku)\,dz\quad (|A^{-1}|=\det A^{-1})$$

and we seek the control u maximizing this expression in the set $U^k=U\times\ldots\times U$.

Theorem 2. Le the control $u_0\in U^k$ deliver the maximum probability of successful search after k steps. Then

$$\left(A_k^T|A^{-1}|^k\int\limits_M\nabla_xp_0((A^{-1})^kz-A_ku_0)\,dz,\Delta u\right)\geqslant 0 \tag{2.2}$$
$$\forall\Delta u\in K_{u^k}(u_0)$$

The proof is analogous to that of Theorem 1.

The problem of search in a game situation is of interest when the target moves under the influence of two controls, u and v: $z_{k+1}=Az_k+u_k+v_k$ and the player with control v, $v\in V$ available to him tries to prevent the encounter of the target with the set M. The theorem remains valid in this case, but the integration in inequality (2.2) must now be carried out over the set $M \overset{*}{-} A_kV^k$ where $\overset{*}{-}$ is the operation of geometrical subtraction of sets /6/ under the assumption that the set $M\overset{*}{-}A_kV^k$ is non-empty and V^k is a convex compactum.

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SYNTHESIS OF THE OPTIMAL CONTROL FOR A LINEAR SYSTEM WITH TWO PHASE CONSTRAINTS*

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The synthesis of a control for a system described by a linear, second-order differential equation with constant coefficients (an oscillatory section) and when two constraints are imposed on the phase coordinates (one of them mixed) is given. The properties of the optimal phase trajectories are described.

1. Formulation of the problem. The following problem arises when constructing the servos for measuring systems. The measuring system intended for tracking an external object is initially given the angular elevation of the object. On receiving the signal, the servo of the system turns its sighting beam in the prescribed direciton. The fastest possible rate of sweep of the sighting beam must be ensured, taking into account the restriction imposed on its rate of motion and on the maximum power demand allowed.

Using this formulation, we will separate the problem of synthesizing the optimal response control $\bar{u}^\circ$ transforming the system

$$dy/dt = -2\xi/Ty - \varphi/T^2 + \bar{u}/T^2, \quad d\varphi/dt = y \qquad (1.1)$$
$$T > 0, \quad 0 < \xi < 1$$

from the arbitrary admissible points φ, y to the origin of coordinates, with the following constraints imposed on the control $\bar{u}$ and phase coordinates:

$$|\bar{u}| \leqslant \bar{u}_0, \quad |y| \leqslant 2\bar{y}_0, \quad |y dy/dt| \leqslant 4\bar{P}_0 \qquad (1.2)$$

(the second condition describes the velocity constraint and the third the power constraint).

Next we consider the case when $\xi \notin (0.519 \div \sqrt{2}/2)$ (see Sect.5).

We know /1/ that the form of the optimal trajectories sought depends, under the constraints given in (1.2), mainly on the form of the roots of the characteristic equation (1.1)

$$\lambda_{1,2} = \varkappa \pm \mu i, \quad \varkappa = -\xi/T < 0, \quad \mu = (1 - \xi^2)^{1/2}/T$$

Transforming the variables

$$\left\|\begin{matrix} z \\ \delta \end{matrix}\right\| = \left\|\begin{matrix} 1/2 & 0 \\ 1/2\varkappa\mu^{-1} & -A \end{matrix}\right\| \left\|\begin{matrix} y \\ \varphi \end{matrix}\right\|; \quad \left(A = \frac{\varkappa^2 + \mu^2}{2\mu}\right) \qquad (1.3)$$

we reduce system (1.2) and the constraints (1.2) to a form suitable for our investigation

$$dz/dt = \varkappa z + \mu\delta + \mu u, \quad d\delta/dt = -\mu z + \varkappa\delta + \varkappa u \qquad (1.4)$$
$$|u| \leqslant u_0, \qquad u_0 = A\bar{u}_0 \qquad (1.5)$$
$$|z| \leqslant y_0, \qquad y_0 = \bar{y}_0/2 \qquad (1.6)$$

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